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STUDY OF THE UNSTEADY TEMPERATURE FIELD IN A SPHERICAL BODY USING CHEBYSHEV—LAGUERRE POLYNOMIALS

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We discuss a method of solving axisymmetric boundary-value problems for the parabolic heat equation in spherical coordinates based on the use of Chebyshev—Laguerre polynomials.

The unsteady heat conduction of a spherical body subject to nonuniform axisymmetric heating of its surface reduces to the solution of a boundary-value problem for the parabolic heat equation. The Laplace transform in time leads to significant computational difficulties in this case. We discuss a new method of finding the unsteady temperature field in a spherical body subject to local heating. The method is based on the use of Chebyshev—Laguerre polynomials [1].

1. Consider a hollow sphere and define spherical coordinates (r, θ, φ) in the usual way. The outer and inner surfaces of the sphere are subject to heat exchange according to Newton's law into media with temperatures $T_c^{\pm}(\theta, F)$, respectively.

The temperature field $T(\gamma, \theta, F)$ inside the sphere is found by solving the following axisymmetric mixed initial-value—boundary-value problem:

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{\partial}{\partial\gamma}\left[(1+\varepsilon\gamma)^2\frac{\partial T}{\partial\gamma}\right]+\frac{\varepsilon^2}{(1+\varepsilon\gamma)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=\frac{\partial T}{\partial F},\quad -1\leqslant\gamma\leqslant 1,\quad 0\leqslant\theta\leqslant\pi, \tag{1}$$

$$\left(\frac{\partial T}{\partial\gamma}\right)^{\pm}\pm \mathrm{Bi}^{\pm}[T^{\pm}-T_c^{\pm}(F,\theta)]=0,\qquad \gamma=\pm 1, \tag{2}$$

$$T(\gamma,\theta,F)=0,\qquad F\leqslant 0, \tag{3}$$

where $r = R(1 + \varepsilon\gamma)$ is the radius of the sphere, $\varepsilon = h/R$; $F = a\tau/h^2$ is the Fourier number; $\mathrm{Bi}^{\pm} = \alpha^{\pm}h/\lambda_t$ are the Biot numbers on the surfaces $\gamma = \pm 1$.

The integral formula

$$T_{nm}(\gamma)=\left(m+\frac{1}{2}\right)\int_0^\infty \exp(-\lambda F)\,L_n(\lambda F)\left[\int_0^\pi T(\gamma,\theta,F)\,P_m(\cos\theta)\sin\theta\,d\theta\right]dF,\quad n,\ m=\overline{0,\ \infty}, \tag{4}$$

defines a double integral transform of the function $T(\gamma, \theta, F)$, where $L_n(\lambda F)$ are the orthogonal Chebyshev—Laguerre polynomials; $P_m(\cos\theta)$ are the orthogonal Legendre polynomials [2]; λ is a positive parameter which we call the regularization parameter.

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$$T_n(\gamma,\ \theta) = \sum_{m=0}^{\infty} T_{nm}(\gamma)\, P_m(\cos\theta), \quad n = \overline{0,\ \infty}, \tag{5}$$

then the series

$$T(\gamma,\ \theta,\ F) = \lambda \sum_{n=0}^{\infty} T_n(\gamma,\ \theta)\, L_n(\lambda F) \tag{6}$$

serves as an inversion formula of the integral transform (4). Applying the integral transform (4) to the problem (1)-(3), we obtain the problem

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{d}{d\gamma}\left[(1+\varepsilon\gamma)^2\frac{d}{d\gamma}T_{nm}\right] - \left[\lambda + \frac{\varepsilon^2 m(m+1)}{(1+\varepsilon\gamma)^2}\right]T_{nm} = \lambda\sum_{k=0}^{n-1}T_{km}, \tag{7}$$

$$\left(\frac{dT_{nm}}{d\gamma}\right)^{\pm} \pm \mathrm{Bi}^{\pm}\,[T_{nm}^{\pm} - T_{cmn}^{\pm}] = 0, \qquad \gamma = \pm 1. \tag{8}$$

The general solution of the triangular system of ordinary differential equations (7) can be written in the form [3]

$$T_{nm}(\gamma) = \sum_{j=0}^{n}[A_{n-j}^{m}G_{jm}(\gamma) + B_{n-j}^{m}W_{jm}(\gamma)], \tag{9}$$

where A_{n-j}^{m}, B_{n-j}^{m} are constants which can be determined from the boundary conditions (8). In view of the structure of the general solution (9), the algebraic system of equations for the arbitrary constants A_{n-j}^{m} and B_{n-j}^{m} will always have the form of a triangular matrix. The functions $G_{jm}(\gamma)$ and $W_{jm}(\gamma)$ are linearly independent particular solutions of the system (7). Using the following easily verified identities for the modified spherical Bessel functions

$$\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right) - 1 - \frac{m(m+1)}{x^2}\right][x^j\, \mathrm{k}_{m-j}(x)] = -2jx^{j-1}\,\mathrm{k}_{m-j+1}(x),$$
$$\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right) - 1 - \frac{m(m+1)}{x^2}\right][x^j\, \mathrm{i}_{m-j}(x)] = 2jx^{j-1}\,\mathrm{i}_{m-j+1}(x), \tag{10}$$

(where $x = \frac{\sqrt{\lambda}}{\varepsilon}(1+\varepsilon\gamma)$; $\mathrm{k}_n(x)$, $\mathrm{i}_n(x)$ are the modified spherical Bessel functions [4]), the functions G_{jm} and W_{jm} can be represented in the form

$$G_{jm}(\gamma) = \sum_{k=1}^{j} a_{jk}^{m} x^k\, \mathrm{i}_{m-k}(x), \quad G_{0m}(\gamma) = a_{00}^{m}\, \mathrm{i}_m(x),$$
$$W_{jm}(\gamma) = \sum_{k=1}^{j} b_{jk}^{m} x^k\, \mathrm{k}_{m-k}(x), \quad W_{0m}(\gamma) = b_{00}^{m}\, \mathrm{k}_m(x). \tag{11}$$

Substitution of these solutions into system (7) reduces the problem to a set of recursion relations for the constants a_{jk}^{m} and b_{jk}^{m} (using the method of undetermined parameters):

$$a_{j1}^{m} = \frac{1}{2}a_{00}^{m}, \quad j = \overline{1,\ n};$$
$$a_{j,k+1}^{m} = \frac{1}{2(k+1)}\sum_{l=k}^{j-1}a_{lk}^{m}, \quad j = \overline{2,\ n}, \quad k = \overline{1,\ j-1};$$
$$b_{j1}^{m} = -\frac{1}{2}b_{00}^{m}, \quad j = \overline{1,\ n};$$
$$b_{j,k+1}^{m} = -\frac{1}{2(k+1)}\sum_{l=k}^{j-1}b_{lk}^{m}, \quad j = \overline{2,\ n}, \quad k = \overline{1,\ j-1}, \tag{12}$$

where the arbitrary constants a_{00}^m and b_{00}^m are determined from the normalization conditions. We note that when $a_{00}^m = b_{00}^m$

$$b_{jk}^m = (-1)^k a_{jk}^m, \quad j = \overline{1, n}, \quad k = \overline{1, j}, \quad j \geqslant k.$$

2. When the heating of the spherical body is uniform (the centrally symmetric case) the temperature field T(γ, F) is determined by the following boundary-value problem:

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{\partial}{\partial\gamma}\left[(1+\varepsilon\gamma)^2\frac{\partial T}{\partial\gamma}\right] = \frac{\partial T}{\partial F}, \tag{13}$$

$$\left(\frac{\partial T}{\partial\gamma}\right)^{\pm} \pm \mathrm{Bi}^{\pm}\left[T^{\pm} - T_c^{\pm}(F)\right] = 0, \qquad \gamma = \pm 1, \tag{14}$$

$$T(\gamma, F) = 0, \qquad F \leqslant 0, \tag{15}$$

where $T_c^{+}(F) = t_0 S_{+}(F),\ T_c^{-}(F) = 0,\ t_0 = \text{const}.$

From (4) with m = 0

$$T_n(\gamma) = \int_0^\infty T(\gamma, F)\exp(-\lambda F) L_n(\lambda F)\, dF \tag{16}$$

is the solution of the boundary-value problem

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{d}{d\gamma}\left[(1+\varepsilon\gamma)^2\frac{d}{d\gamma}T_n\right] - \lambda T_n = \lambda\sum_{k=0}^{n-1} T_k, \tag{17}$$

$$\left(\frac{dT_n}{d\gamma}\right)^{\pm} \pm \mathrm{Bi}^{\pm} T_n^{\pm} = \begin{cases} \dfrac{t_0 \mathrm{Bi}^{+} \delta_{0n}}{\lambda}, & \gamma = +1, \\ 0, & \gamma = -1, \end{cases} \tag{18}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{19}$$

is the Kronecker delta.

With the help of (9) the general solution of (17) takes the form

$$T_n(\gamma) = \sum_{j=0}^{n}\left[A_{n-j} G_j(\gamma) + B_{n-j} W_j(\gamma)\right], \tag{19}$$

where A_{n-j} and B_{n-j} are constants determined from the boundary conditions (18), and the functions $G_j(\gamma)$ and $W_j(\gamma)$ can be represented in the following form, with the help of (11):

$$G_j(\gamma) = \sum_{k=1}^{j} a_{jk} x^k \mathrm{i}_{-k}(x), \quad G_0(\gamma) = a_{00}\,\mathrm{i}_0(x),$$

$$W_j(\gamma) = \sum_{k=1}^{j} (-1)^k a_{jk} x^k \mathrm{k}_k(x), \quad W_0(\gamma) = a_{00}\,\mathrm{k}_0(x). \tag{20}$$

Here $\mathrm{k}_k(x) = \mathrm{k}_{-k}(x)$ [4], and the constants a_{jk} are determined from recursion relations similar to [12]

$$a_{j1} = \frac{1}{2} a_{00}, \quad j = \overline{1, n};$$

$$a_{j,k+1} = \frac{1}{2(k+1)}\sum_{l=k}^{j-1} a_{lk}, \quad j = \overline{2, n}, \quad k = \overline{1, j-1}, \tag{21}$$

where the arbitrary constants a_{00} is taken to be equal to unity in the calculations below.

The constants A_{n-j} and B_{n-j} can be found by substituting the general solution $T_n(\gamma)$ of [17] (given by (19)) into the boundary conditions (18).

For n = 0 the problem (13)-(15) has the solution

$$T_0(\gamma)=\frac{t_0\mathrm{Bi}^+}{\lambda\Delta}\{[\sqrt{\lambda}\,\mathrm{k}_0'(x^-)-\mathrm{Bi}^-\mathrm{k}_0(x^-)]\,\mathrm{i}_0(x)-[\sqrt{\lambda}\,\mathrm{i}_0'(x^-)-\mathrm{Bi}^-\mathrm{i}_0(x^-)]\,\mathrm{k}_0(x)\},\tag{22}$$

where $\Delta=[\sqrt{\lambda}\,\mathrm{i}_0'(x^+)+\mathrm{Bi}^+\mathrm{i}_0(x^+)][\sqrt{\lambda}\,\mathrm{k}_0'(x^-)-\mathrm{Bi}^-\mathrm{k}_0(x^-)]-[\sqrt{\lambda}\,\mathrm{k}_0'(x^+)+\mathrm{Bi}^+\mathrm{k}_0(x^+)][\sqrt{\lambda}\,\mathrm{i}_0'(x^-)-\mathrm{Bi}^-\mathrm{i}_0(x^-)]$; $x^\pm=\sqrt{\lambda}/\varepsilon(1\pm\varepsilon)$; $'$ — denotes the derivative with respect to x.

For n = 1, 2, ..., we substitute (19) into the boundary conditions (18) and obtain a system of algebraic equations for the constants A_{n-j} and B_{n-j}. Successive subtraction of this system of equations reduces it to a system of algebraic equations for the constants A_n and B_n:

$$\begin{aligned}&A_n[\sqrt{\lambda}\,\mathrm{i}_0'(x^+)+\mathrm{Bi}^+\mathrm{i}_0(x^+)]+B_n[\sqrt{\lambda}\,\mathrm{k}_0'(x^+)+\mathrm{Bi}^+\mathrm{k}_0(x^+)]=f_n^+\quad(\gamma=+1),\\&A_n[\sqrt{\lambda}\,\mathrm{i}_0'(x^-)-\mathrm{Bi}^-\mathrm{i}_0(x^-)]+B_n[\sqrt{\lambda}\,\mathrm{k}_0'(x^-)-\mathrm{Bi}^-\mathrm{k}_0(x^-)]=f_n^-\quad(\gamma=-1),\end{aligned}\tag{23}$$

where

$$f_n^\pm=-\sum_{j=1}^{n}\left\{A_{n-j}\left[\left(\frac{dG_j}{d\gamma}\right)^\pm\pm\mathrm{Bi}^\pm G_j^\pm\right]+B_{n-j}\left[\left(\frac{dW_j}{d\gamma}\right)^\pm\pm\mathrm{Bi}^\pm W_j^\pm\right]\right\}.$$

With the values of the coefficients A_n and B_n found from system (23), the solution of problem (13)-(15) is determined from (6) by the following series

$$\Theta(\gamma,F)=\frac{\lambda}{t_0}\sum_{n=0}^{\infty}\sum_{j=0}^{n}[A_{n-j}G_j(\gamma)+B_{n-j}W_j(\gamma)]\,L_n(\lambda F),\ \Theta(\gamma,F)=\frac{T(\gamma,F)}{t_0}.\tag{24}$$

3. We now find the solution of the problem (13)-(15) using finite integral transforms. The function $T(\gamma, F)$ is written as a sum

$$T(\gamma,F)=T_1(\gamma)+T_2(\gamma,F).\tag{25}$$

Here $T_1(\gamma)$ is the solution of the steady-state problem

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{d}{d\gamma}\left[(1+\varepsilon\gamma)^2\frac{dT_1}{d\gamma}\right]=0,\tag{26}$$

$$\left(\frac{dT_1}{d\gamma}\right)^\pm\pm\mathrm{Bi}^\pm(T_1^\pm-T_c^\pm)=0\quad(\gamma=\pm1),\tag{27}$$

and has the form

$$T_1(\gamma)=\frac{t_0\mathrm{Bi}^+(1+\varepsilon)[1+B^-(1+\gamma)]}{(1+\varepsilon\gamma)(B^++B^-+2B^+B^-)},\tag{28}$$

where $B^\pm=\mathrm{Bi}^\pm\mp\dfrac{\varepsilon}{1\pm\varepsilon}$.

The function $T_2(\gamma, F)$ is the solution of the nonsteady problem

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{\partial}{\partial\gamma}\left[(1+\varepsilon\gamma)^2\frac{\partial T_2}{\partial\gamma}\right]=\frac{\partial T_2}{\partial F},\tag{29}$$

$$\left(\frac{\partial T_2}{\partial\gamma}\right)^\pm\pm\mathrm{Bi}^\pm T_2^\pm=0\quad(\gamma=\pm1),\tag{30}$$

$$T_2(\gamma,0)=-T_1(\gamma).\tag{31}$$

After performing the substitution $T_i(\gamma, F) = t_i(\gamma, F)/(1+\varepsilon\gamma)$ (i = i, 2) we obtain

$$\frac{\partial^2 t_2}{\partial \gamma^2} = \frac{\partial t_2}{\partial F}, \tag{32}$$

$$\left(\frac{\partial t_2}{\partial \gamma}\right)^{\pm} \pm B^{\pm} t_2^{\pm} = 0 \qquad \left(\gamma = \pm 1\right), \tag{33}$$

$$t_2(\gamma, 0) = -t_1(\gamma), \tag{34}$$

whose solution can be found by reducing it to a Cauchy problem with the help of a finite integral transform with respect to the variable γ [5].

As the kernel of the finite integral transform in γ, we choose the orthonormal system of functions

$$\varphi_n(\gamma) = \alpha_n \cos(\lambda_n \gamma + \delta_n), \tag{35}$$

which are the eigenfunctions of the Sturm–Liouville problem

$$\begin{gathered} \varphi_n'' + \lambda_n^2 \varphi_n = 0, \\ (\varphi_n')^{\pm} \pm B^{\pm}\varphi_n^{\pm} = 0 \qquad \left(\gamma = \pm 1\right), \\ \alpha_n = \sqrt{\frac{2\lambda_n(1+\eta^2)}{2\lambda_n + \sin 2\lambda_n + \eta^2(2\lambda_n - \sin 2\lambda_n)}}, \\ \eta = \frac{\lambda_n \sin\lambda_n - B^- \cos\lambda_n}{\lambda_n \cos\lambda_n + B^- \sin\lambda_n}, \quad \sin\delta_n = -\frac{(B^- \operatorname{ctg}\lambda_n - \lambda_n)\sin\lambda_n}{\sqrt{\lambda_n^2 + (B^-)^2}}, \\ \cos\delta_n = \frac{(\lambda_n \operatorname{ctg}\lambda_n + B^-)\sin\lambda_n}{\sqrt{\lambda_n^2 + (B^-)^2}}, \end{gathered} \tag{36}$$

where λ_n are the roots of the transcendental equation

$$\frac{\operatorname{tg} 2\lambda_n}{\lambda_n} = \frac{B^+ + B^-}{\lambda_n^2 - B^+ B^-}, \tag{37}$$

then the function $t_2(\gamma, F)$ can be represented as the series:

$$t_2(\gamma, F) = \sum_{n=0}^{\infty} t_{2,n}(F)\varphi_n(\gamma), \tag{38}$$

and

$$t_{2,n}(F) = \int_{-1}^{1} t_2(\gamma, F)\varphi_n(\gamma)\,d\gamma. \tag{39}$$

With the help of (39), (32), and the initial conditions (34) we find

$$\begin{gathered} \frac{dt_{2,n}}{dF} + \lambda_n^2 t_{2,n} = 0, \\ t_{2,n}(0) = -t_{1,n} = \text{const}. \end{gathered} \tag{40}$$

The solution of the Cauchy problem (40) has the form

$$t_{2,n}(F) = -\frac{2t_0 \mathrm{Bi}^+(1+\varepsilon)}{B^+ + B^- + 2B^+B^-}\left[(1+B^-)A_n \sin\lambda_n - B^- B_n\left(\cos\lambda_n - \frac{\sin\lambda_n}{\lambda_n}\right)\right]\frac{\exp(-\lambda_n^2 F)}{\lambda_n}, \tag{41}$$

where

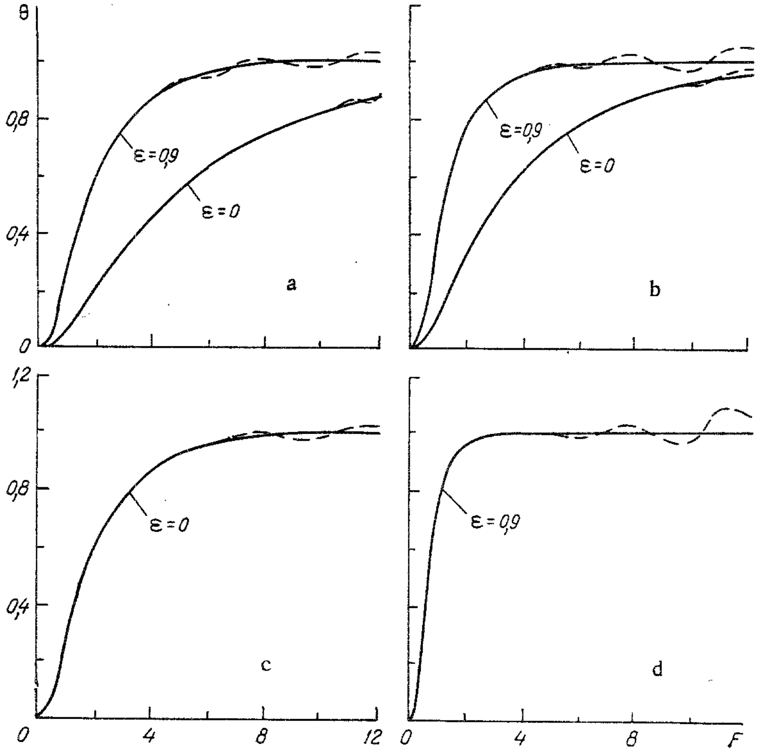

Fig. 1. Dependence of the dimensionless temperature Θ on the Fourier number F for different Bi^+ and ε: a) $Bi^+ = 0.5$; b) $Bi^+ = 1$, c) $Bi^+ = 10$, d) $Bi^+ = 10$.

$$A_n = \sqrt{\frac{2\lambda_n}{2\lambda_n + \sin 2\lambda_n + \eta^2(2\lambda_n - \sin 2\lambda_n)}}, \quad B_n = -\eta A_n. \tag{41}$$

Hence substituting (41) into (38), we obtain the solution of the problem (32)-(34) and therefore the required temperature field of the problem (13)-(15) takes the form

$$\Theta(\gamma, F) = \frac{\mathrm{Bi}^+(1+\varepsilon)}{1+\varepsilon\gamma}\left\langle \frac{1+B^-(1+\gamma)}{B^+ + B^- + 2B^+B^-} - 2\sum_{n=1}^{\infty} \frac{\{\lambda_n \cos[\lambda_n(1+\gamma)] + B^-\sin[\lambda_n(1+\gamma)]\}\exp(-\lambda_n^2 F)}{\lambda_n\{\lambda_n[1+2(B^+ + B^-)] + B^+B^-/\lambda_n\}\sin 2\lambda_n + 2\lambda_n(\lambda_n^2 - B^+B^-)\cos 2\lambda_n}\right\rangle, \tag{42}$$

where $\Theta(\gamma, F) = T(\gamma, F)/t_0$.

We note that the solution of (13)-(15) found with the help of the Laplace transform can also be written in the form (42).

4. The solution was computed numerically using (24) and (42) for $\gamma = -1$, $Bi^- = 0$ (corresponding to a thermally insulated inener surface for the sphere), $Bi^+ = 0.5, 1, 10$. The results are shown in Fig. 1, where the dashed curves show the temperature found by the Chebyshev–Laguerre polynomial method, and the solid curves show the temperature found using finite integral transforms or the Laplace transform. The regularization parameter λ was chosen to be unit and the series (24) and (42) were cut off at n = 30 and n = 10, respectively. We note that by adjusting the regularization parameter, it is possible to obtain the same temperature field by the different methods over the entire interval of the time variable, so long as the argument of the functions $L_n(\lambda F)$ satisfies the condition $\lambda F \leqslant 5$. It can also be shown that when the relative thickness satisfies $\varepsilon \leqslant 0.03$, the temperature field inside a spherical body is the same as that inside a layer of thickness 2h (ε = 0) for the same initial and boundary conditions to within an error of no more than 5%.

NOTATION

T, temperature; r, θ, φ, spherical coordinates; τ, time; R, mean radius; h, half-thickness of the spherical wall; γ, dimensionless coordinate normal to the surface of the spheres; a, λ_t, thermal diffusivity and thermal conductivity, respectively; $\alpha^{\pm}$, coefficients of heat exchange with the surfaces $\gamma = \pm 1$; $T_c^{\pm}$ temperatures of the external media interacting with the surfaces $\gamma = \pm 1$; $\Theta(\gamma, F)$, dimensionless temperature; t_0, initial temperature.

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